Sains Malaysiana 40(10)(2011): 1173–1178

Improved Sufficient Conditions for Monotonic Piecewise Rational Quartic Interpolation

(Syarat Cukup yang Lebih Baik untuk Interpolasi Kuartik Nisbah Cebis Demi Cebis Berekanada)

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ABSTRACT

In 2004, Wang and Tan described a rational Bernstein-Bézier curve interpolation scheme using a quartic numerator and linear denominator. The scheme has a unique representation, with parameters that can be used either to change the shape of the curve or to increase its smoothness. Sufficient conditions are derived by Wang and Tan for preserving monotonicity, and for achieving either C^1 or C^2 continuity. In this paper, improved sufficient conditions are given and some numerical results presented.

Keywords: Continuity; interpolation; monotonicity; rational Bernstein-Bézier

ABSTRAK

Pada tahun 2004, Wang dan Tan telah memerikan suatu skema interpolasi lengkung Bernstein-Bézier nisbah menggunakan pembilang kuartik dan penyebut linear. Skema tersebut mempunyai suatu perwakilan yang unik, dengan parameter yang boleh digunakan untuk menukar sama ada bentuk lengkung atau untuk meningkatkan kelicinan lengkung. Syarat cukup diterbitkan oleh Wang dan Tan untuk mengekalkan keekanadaan, dan untuk mencapai keselanjaran sama ada C^1 atau C^2. Dalam kertas kerja ini, syarat perlu yang lebih baik dan beberapa keputusan berangka diberikan.

Kata kunci: Interpolasi; keselanjaran; keekanadaan; Bernstein-Bézier nisbah

INTRODUCTION

Wang and Tan (2004) construct a rational curve interpolant which matches given data while preserving the monotonic property of the interpolated data. They use a Bernstein-Bézier quintic rational interpolant with quartic numerator and linear denominator, producing piecewise curves with either C^1 or C^2 continuity. Each curve segment has a shape parameter, but the authors do not make it clear how the values of these parameters are chosen. Their approach is the same as that of Duan et al. (1999), but uses a different polynomial degree for the numerator. They derive sufficient conditions on the first derivative at each of the given data points to ensure monotonicity is preserved.

Motivated by their study, we are proposing more relaxed sufficient conditions and we will show how interactive adjustment of either the shape parameters or the first derivatives can ensure C^2 continuity. Other useful properties of the method by Wang and Tan (2004) include the following: no additional knots are needed; unlike the schemes of Duan et al. (1999), Gregory & Delbourgo (1982) and Sarfraz (2000), it does not require the solution to a system of equations to ensure C^2 continuity; and it is a local scheme.

THE RATIONAL BERNSTEIN-BÉZIER QUINTIC INTERPOLANT

Suppose $\{(x_i, f_i), i = 1,\ldots,n\}$ is a given set of data points, where $x_1 < x_2 < \ldots < x_n$ and $f_1, f_2, \ldots, f_n$ are real numbers. Suppose also that $h_i = x_{i+1} - x_i$, $\Delta_i = \dfrac{(f_{i+1} - f_i)}{h_i}$, $i = 1, \ldots, n-1$. For $x \in [x_i, x_{i+1}]$, $i = 1, \ldots, n-1$, we define a local variable θ by $\theta = \dfrac{(x - x_i)}{h_i}$ i.e. $0 \le \theta \le 1$.

Wang and Tan (2004) define an interpolating curve $s(x)$ on $[x_1, x_n]$. On each interval $[x_i, x_{i+1}]$, $i = 1, \ldots, n-1$, $s(x)$ is defined as:

$$s(x) = s(x_i + h_i\theta) \equiv S_i(\theta) = \frac{P_i(\theta)}{Q_i(\theta)}, \quad i = 1, \ldots, n-1, \qquad (1)$$

where $P_i(\theta) = \alpha_i f_i(1-\theta)^4 + V_{i1}(1-\theta)^3\theta + V_{i2}(1-\theta)^2\theta^2 + V_{i3}(1-\theta)\theta^3 + \beta_i f_{i+1}\theta^4$, $Q_i(\theta) = \alpha_i(1-\theta) + \beta_i\theta$, $V_{i1} = (3\alpha_i + \beta_i) f_i + \alpha_i h_i d_i$, $V_{i2} = 3\alpha_i f_{i+1} + 3\beta_i f_i$, $V_{i3} = (\alpha_i + 3\beta_i) f_{i+1} - \beta_i h_i d_{i+1}$. Note that the numerator $P_i(\theta)$ is a quartic Bernstein-Bézier polynomial and the total degree of $s(x) \equiv S_i(\theta)$ is 5. α_i, β_i are non-zero shape parameters such that $\text{sign}(\alpha_i) = \text{sign}(\beta_i)$ (Note: Wang & Tan (2004) just choose them to be positive). Thus, the denominator of (1) is non-zero. $d_i \ge 0$ denotes a given (or an estimated) value for the first derivative at x_i. By defining $e_i = \dfrac{\alpha_i}{\beta_i} > 0$, $i = 1, \ldots, n-1$, (1) can be expressed in terms of the single shape parameter, e_i:

$$\left.\begin{aligned}
&P_i(\theta)=e_i f_i(1-\theta)^4+W_{i1}(1-\theta)^3\theta+W_{i2}(1-\theta)^2\theta^2+W_{i3}(1-\theta)\theta^3+f_{i+1}\theta^4,\\
&Q_i(\theta)=e_i(1-\theta)+\theta,\\
&W_{i1}=(3e_i+1)f_i+e_ih_id_i,\\
&W_{i2}=3e_if_{i+1}+3f_i,\\
&\text{and } W_{i3}=(e_i+3)f_{i+1}-h_id_{i+1}.
\end{aligned}\right\} \quad (2)$$

Unless stated otherwise, this form of the interpolant will be used throughout the remainder of this paper.

It is clear from (2) that $s(x)$ satisfies

$$s(x_i)=f_i,\ s(x_{i+1})=f_{i+1},\ s'(x_i)=d_i, \text{ and } s'(x_{i+1})=d_{i+1}. \quad (3)$$

Hence $s(x)\in C^1[x_1,x_n]$.

MONOTONICITY-PRESERVING INTERPOLATION

Suppose $s(x)$ is defined according to (1) and (2). We assume that the data are monotonic increasing, so that $f_1\le f_2\le \dots \le f_n$ or equivalently

$$\Delta_i>0,\ i=1,\dots,n-1. \quad (4)$$

We assume that the first derivatives d_i, $i=1,\dots,n$ have been given as part of the data, or are calculated from the given data, so that

$$d_i\ge 0,\ i=1,\dots,n. \quad (5)$$

$s(x)$ is monotone if $s'(x)\ge 0$ for all $x\in[x_i,x_{i+1}]$, $i=1,\dots,n-1$. After some simplifications, Wang & Tan (2004) write:

$$s'(x)=h_i^{-1}S'(\theta)=\frac{\sum_{j=0}^{4}A_{ij}(1-\theta)^{4-j}\theta^j}{[\alpha_i(1-\theta)+\beta_i\theta]^2}=\frac{N_i(\theta)}{[\alpha_i(1-\theta)+\beta_i\theta]^2}, \quad (6)$$

where

$A_{i0}=\alpha_i^2d_i$, $A_{i1}=2\alpha_i^2(3\Delta_i-d_i)$, $A_{i2}=3\alpha_i\beta_i(4\Delta_i-d_i-d_{i+1})$, $A_{i3}=2\beta_i^2(3\Delta_i-d_{i+1})$, and $A_{i4}=\beta_i^2d_{i+1}$.

The denominator in (6) is always positive. Therefore, considering the numerator in (6), Wang & Tan (2004) conclude that $s'(x)\ge 0$ if:

$$d_i\le 3\Delta_i,\ d_{i+1}\le 3\Delta_i,\ d_i+d_{i+1}\le 4\Delta_i, \quad (7)$$

are satisfied (Figure 1). Thus, they state sufficient conditions for a monotone interpolant as follows:

PROPOSITION 1 *Given a monotonic increasing set of data satisfying (4) and (5), there exists a class of monotonic rational (quartic/linear) interpolating splines* $s(x)\in C^{(1)}[a,b]$ *involving the parameters* α_i, β_i, *provided that (7) holds.*

(6) can be expressed in terms of e_i:

$$s'(x)=h_i^{-1}S'(\theta)=\frac{\sum_{j=0}^{4}A_{ij}(1-\theta)^{4-j}\theta^j}{[e_i(1-\theta)+\theta]^2}=\frac{N_i(\theta)}{[e_i(1-\theta)+\theta]^2}, \quad (8)$$

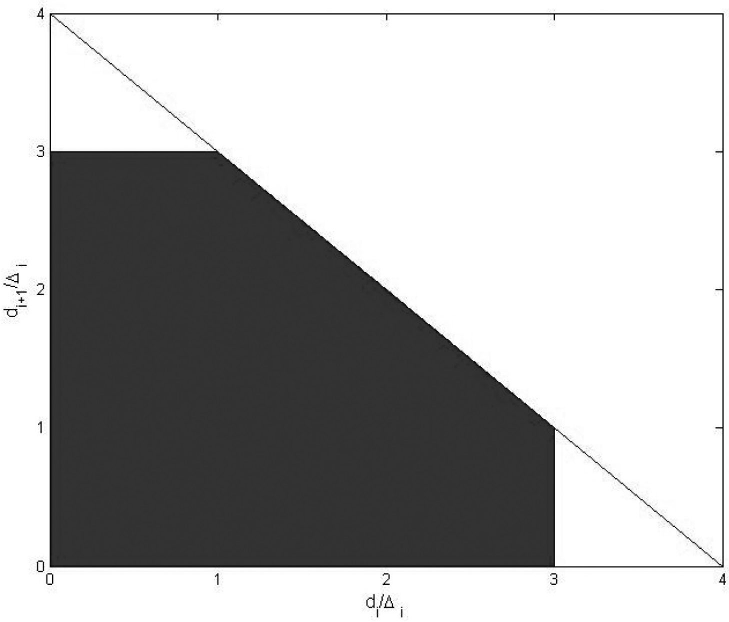

FIGURE 1. Monotonicity region of Wang and Tan (2004)

where

$A_{i0} = e_i^2 d_i$, $A_{i1} = 2e_i^2(3\Delta_i - d_i)$, $A_{i2} = 3e_i(4\Delta_i - d_i - d_{i+1})$,

$A_{i3} = 2(3\Delta_i - d_{i+1})$, and $A_{i4} = d_{i+1}$.

Suppose $x \in [x_i, x_{i+1}]$ and $d_i = n_1\Delta_i$, $d_{i+1} = n_2\Delta_i$, $0 \le n_1, n_2 \le 3$. From (8) we have,

$$\left.\begin{aligned} N_i(\theta) = \Delta_i(&e_i^2 n_1(1-\theta)^4 + n_2\theta^4 + (1-\theta)\theta[2e_i^2(3-n_1)(1-\theta)^2 \\ &+3e_i(4-n_1-n_2)\theta(1-\theta) + 2(3-n_2)\theta^2]). \end{aligned}\right\} \tag{9}$$

Omitting Δ_i which is non-negative and common to all terms, and after some straightforward algebraic manipulation, (9) becomes:

$(e_i n_1(1-\theta)^2 - n_2\theta^2)(e_i(1-\theta)^2 - \theta^2) + 2(1-\theta)\theta(e_i(1-\theta) + \theta)[e_i(1-\theta)(3-n_1) + \theta(3-n_2)] \ge (e_i n_1(1-\theta)^2 - n_2\theta^2)(e_i(1-\theta)^2 - \theta^2)$ since $n_1, n_2 \le 3$.

Now suppose $n_1 \ge n_2$. It follows that:

$(e_i n_1(1-\theta)^2 - n_2\theta^2)(e_i(1-\theta)^2 - \theta^2) \ge (e_i n_1(1-\theta)^2 - n_1\theta^2)(e_i(1-\theta)^2 - \theta^2) = n_1(e_i(1-\theta)^2 - \theta^2)^2 \ge 0$,

which then implies that $N_i(\theta) \ge 0$. We may write

$$\left.\begin{aligned} &\left(e_i n_1(1-\theta)^2 - n_2\theta^2\right)\left(e_i(1-\theta)^2 - \theta^2\right) \\ &= \left(n_2\theta^2 - e_i n_1(1-\theta)^2\right)\left(\theta^2 - e_i(1-\theta)^2\right). \end{aligned}\right\} \tag{10}$$

Hence, if $n_2 > n_1$, by applying a similar argument to (10), will give us $N_i(\theta) > 0$. We now have the following proposition, as an improvement to the proposed sufficient conditions in Proposition 1.

PROPOSITION 2 *Suppose a monotonic increasing set of data satisfies (4) and (5). Consider rational splines s(x)* $\in C^1[x_1, x_n]$, *of the form (1), (2) that interpolate these data. These splines preserve the monotonicity of the data for all values of the non-negative shape parameters* e_i, $i = 1, \ldots, n-1$ *if:*

$$\frac{d_i}{\Delta_i} \le 3, \text{ and } \frac{d_{i+1}}{\Delta_i} \le 3, i = 1, \ldots, n-1. \tag{11}$$

The new monotonicity region is displayed in Figure 2.

It is easy to write an algorithm to generate C^1 monotonicity-preserving curves using the result of Proposition 2. An outline is as follows.

OUTLINE OF ALGORITHM

1. Input the number of data points, n and data points $\{x_i, f_i\}_{i=1}^{n}$.
2. For $i = 1, \ldots, n-1$
   a. Define h_i and Δ_i
   b. Initialize d_i so that $0 \le d_1 \le 3\Delta_1$, $0 \le d_n \le 3\Delta_{n-1}$ and $0 \le d_i \le \min(3\Delta_i, 3\Delta_{i-1})$
   c. Initialize $e_i > 0$
   d. Calculate the inner control ordinates W_{i1}, W_{i2}, W_{i3} using (2) and generate the piecewise interpolating curve using (1).

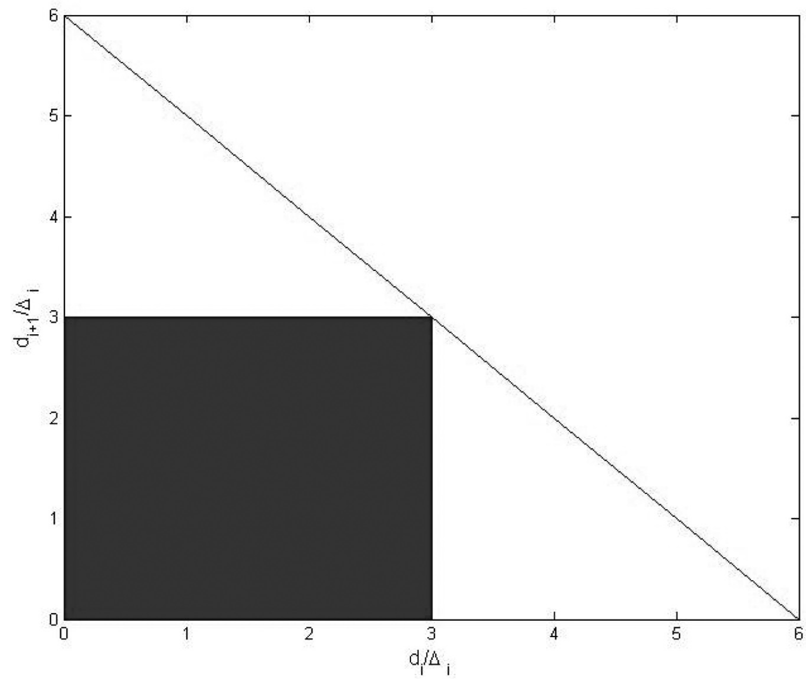

FIGURE 2. Our proposed monotonicity region

3. Until no more changes are necessary and for $i = 1, \ldots, n-1$
   a. Modify, if necessary, d_i (retaining the conditions $0 \le d_1 \le 3\Delta_1$, $0 \le d_n \le 3\Delta_{n-1}$ and $0 \le d_i \le \min(3\Delta_i, 3\Delta_{i-1})$)
   b. Modify, if necessary, e_i (retaining the condition $e_i > 0$)
   c. Calculate the inner control ordinates W_{i1}, W_{i2}, W_{i3} using (2) and generate the piecewise interpolating curve using (1).

Step 2 of the algorithm produces an initial curve that guarantees monotonicity, while step 3 allows a user to repeatedly modify the curve, while still guaranteeing monotonicity, until a visually pleasing curve is obtained.

Note that Wang and Tan (2004) require:

$$d_i = \frac{h_i \Delta_{i-1} + h_{i-1}\Delta_i}{\frac{2}{3}\left(h_{i-1} + h_i\right) + \frac{1}{3}\left(\frac{\beta_i}{\alpha_i} h_{i-1} + \frac{\alpha_{i-1}}{\beta_{i-1}} h_i\right)}, \quad i = 2, \ldots, n-1 \tag{12}$$

for their scheme to be C^2 continuous.

Note that if the derivative values d_i, $i = 2, \ldots, n-1$ are given, then we may rearrange (12) as follows:

$$e_i = \frac{h_{i-1} d_i}{\left(3\Delta_{i-1} - \left(2 + e_{i-1}\right) d_i\right) h_i + \left(3\Delta_i - 2d_i\right) h_{i-1}}, \quad i = 2, \ldots, n-1, \tag{13}$$

thereby giving conditions on e_i, $i = 2, \ldots, n-1$ to ensure C^2 continuity.

An algorithm to generate a C^2 monotonic rational interpolant using (11) and condition (13) is virtually identical to the algorithm above for a C^1 curve. The only difference is that only the first derivative values may be changed by the user, the e_i values must be calculated using either (13), or a suitable choice for e_i.

NUMERICAL EXAMPLES

In order to illustrate our curve interpolation scheme, we will use the same data set in Table 1 from Sarfraz (2000) which was used by Wang and Tan (2004) (Figure 3).

We have also chosen two classical data, the so-called Akima's data set (Akima 1970) and a sigmoidal function, $f(x) = \dfrac{1}{\sqrt{1 + 2e^{-3(x-6,7)}}}$, from Sarfraz (2003) to illustrate our scheme (Figures 4 and 5).

TABLE 1. Sarfraz's data set

i	1	2	3	4	5
x_i	0	6	10	29.5	30
f_i	0.01	15	15	25	30

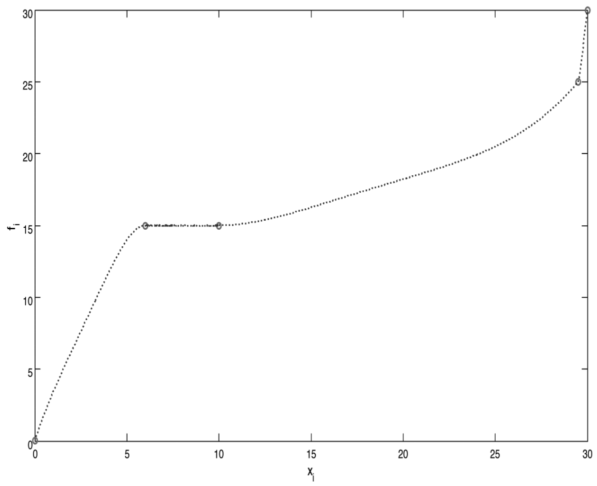

FIGURE 3(a). Monotonicity-preserving interpolant using data in Table 1 and condition (7)

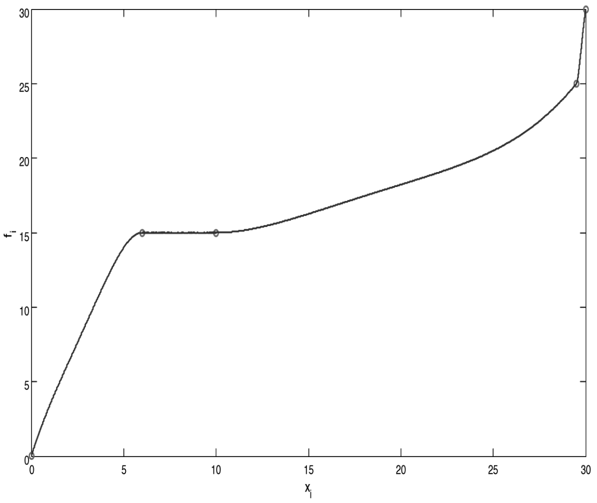

FIGURE 3(b). Monotonicity-preserving interpolant using data in Table 1 and condition (11)

If the end point derivatives values, d_1 and d_n are not given, then they can be estimated using the following two formulas (Delbourgo & Gregory 1985; Hussain & Sarfraz 2009):

Three-point difference approximation (arithmetic mean method)

$$\begin{aligned} d_1 &= \Delta_1 + \left(\Delta_1 - \Delta_2\right)\left(\frac{h_1}{h_1 + h_2}\right), \\ d_n &= \Delta_{n-1} + \left(\Delta_{n-1} - \Delta_{n-2}\right)\left(\frac{h_{n-1}}{h_{n-1} + h_{n-2}}\right). \end{aligned} \tag{14}$$

Note that when d_1 or d_n is negative then its value is set to be 0. Meanwhile, $d_i : i = 2, \ldots, n-1$ are calculated from the C^2 continuity condition (12).

Non-linear approximation (geometric mean method):

$$d_1 = \Delta_1 \left[\Delta_1 \Delta_{3,1} \right]^{h_1/h_2},$$
$$d_n = \Delta_{n-1} \left[\Delta_{n-1} \Delta_{n,n-2} \right]^{h_{n-1}/h_{n-2}}, \tag{15}$$

where

$$\Delta_{3,1} = \frac{f_3 - f_1}{x_3 - x_1}, \; \Delta_{n,n-2} = \frac{f_n - f_{n-2}}{x_n - x_{n-2}}.$$

The approximate values of d_1 and d_n are always positive if we use the geometric mean method.

REMARK. If $\Delta_i = 0$, then it is necessary to set $d_i = d_{i+1} = 0$, so that $s(x) = f_i = f_{i+1}$, a constant on $[x_i, x_{i+1}]$. It should be noted that for the Akima's data set, $s(x)$ is constant in the interval [0,8] and the scheme is only applied over the interval [8,15].

CONCLUSION

In this paper, we improved the monotocity region proposed by Wang and Tan (2004) for a Bernstein-Bézier quintic rational interpolant (with quartic numerator and linear denominator). The resulting curve preserves monotonicity of the data. We also propose an algorithm to generate a C^1 or C^2 curve which preserve the monotonic data. This scheme is local, simple to use, requires few computational steps and the output is comparable to the work of Sarfraz (2000, 2002, 2003).

ACKNOWLEDGEMENTS

The first author would like to extend his gratitude to Universiti Sains Malaysia for supporting this work under its Short-term Research Grant 304/PMATHS/6310027. Much of this work was carried out during his visit to Lincoln

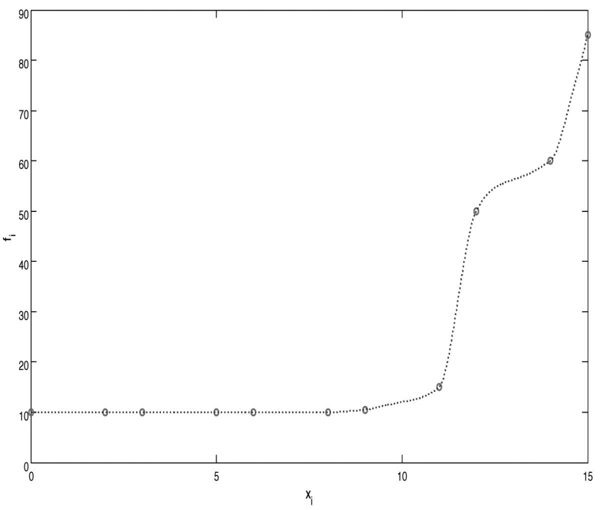

FIGURE 4(a) Monotonicity-preserving interpolant using data in Table 2 and condition (7)

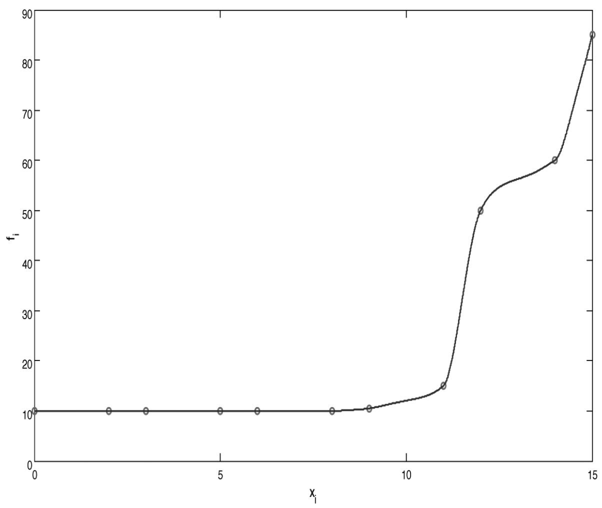

FIGURE 4(b) Monotonicity-preserving interpolant using data in Table 2 and condition (11)

TABLE 2. Akima's data set

i	1	2	3	4	5	6	7	8	9	10	11
x_i	0	2	3	5	6	8	9	11	12	14	15
f_i	10	10	10	10	10	10	10.5	15	50	60	85

TABLE 3. Sigmoidal function on the interval [1,11]

i	1	2	3	4	5	
x_i	1	2	3	4	5	
f_i	0.0001	0.0006	0.0027	0.0123	0.0551	
i	6	7	8	9	10	11
x_i	6	7	8	9	10	11
f_i	0.2402	0.7427	0.9804	0.9990	0.9999	1

University in February 2010. He wishes to acknowledge Lincoln University's financial support.

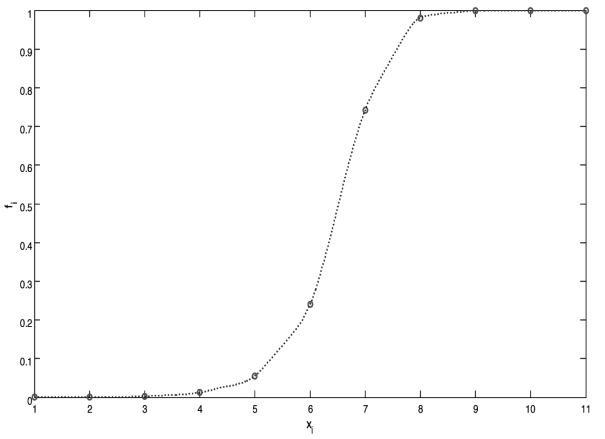

FIGURE 5(a) Monotonicity-preserving interpolant using data in Table 3 and condition (7)

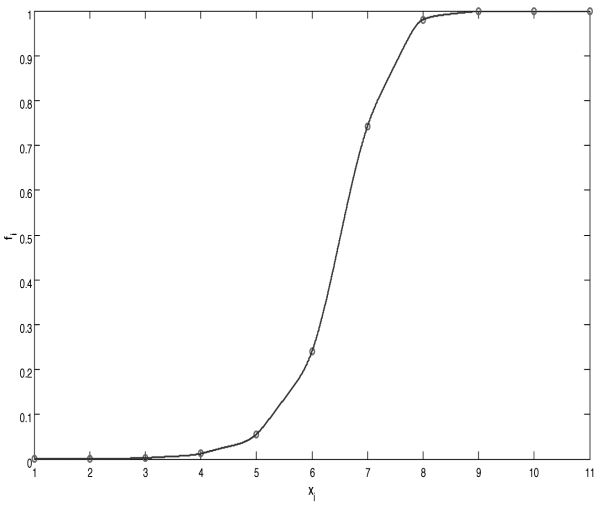

FIGURE 5(b) Monotonicity-preserving interpolant using data in Table 3 and condition (11)

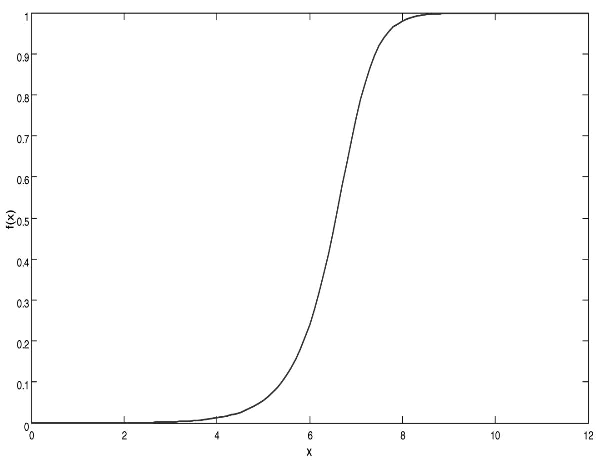

FIGURE 5(c) The actual sigmoidal function,

$$f(x) = \frac{1}{\sqrt{1 + 2e^{-3(x-6.7)}}}$$

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Received: 7 July 2010
Accepted: 17 January 2011